Friday! Recall $z^a := e^{a \log z}$ extended z^n, $z^{p/q}$

$$\frac{d}{dz} z^a = a z^{a-1}$$

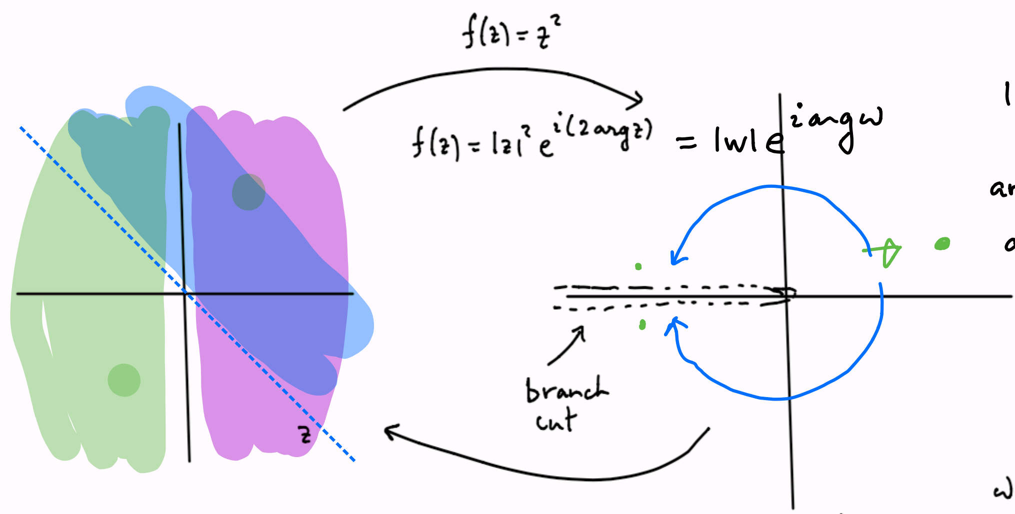
Example 2) $f(z) = z^2$, $g(w) = \sqrt{w}$ (for some branch choice). Note for any branch choice of g,

On Friday we'll do this in depth

$$f(g(w)) = w$$
$$f'(g(w))g'(w) = 1$$
$$\bullet \quad g'(w) = \frac{1}{f'(g(w))} = \frac{1}{2\,g(w)} = \frac{1}{2} w^{-\frac{1}{2}}$$

from Wed.

① Describe the range of the branch of the square root function defined below. Write down two other branch choices - one using the same branch cut, and another one using a different cut. ② ③



$|z|^2 = |w|$
$|z| = |w|^{1/2}$
$\arg w = 2 \arg z + 2\pi k$
$\arg z = \frac{1}{2} \arg w + \pi k$

$g(w) = |w|^{1/2} e^{i\left(\frac{\arg w}{2}\right)}$ on branched domain $-\pi < \arg w < \pi$

$-\pi/2 < \frac{\arg w}{2} < \pi/2$

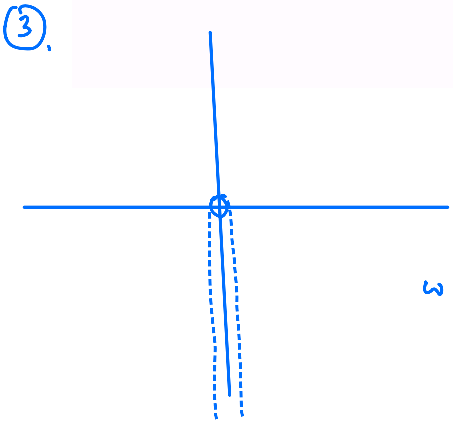
Range $g = \{x+iy \mid x>0\}$ (right half plane)

② $g_2(w) = |w|^{1/2} e^{i\left(\frac{\arg w}{2} + \pi\right)}$ $\qquad e^{i\pi} = -1$

$= -g(w)$ $\qquad -\pi < \arg w < \pi$

Range g_2 : $\pi/2 < \frac{\arg w}{2} + \pi < \frac{3\pi}{2}$.
= left half plane.

③.



$-\pi/2 < \arg w < \frac{3}{2}\pi$

$\sqrt{w} = |w|^{1/2} e^{i\left(\frac{\arg w}{2}\right)}$

Range $g_3(w)$ is everything above line $y = -x$

$-\pi/4 < \frac{\arg w}{2} < \frac{3}{4}\pi$

Example 3) Find a definition and branched domain for

$$f(z)=\sqrt{z^2-1}\,. = (z^2-1)^{1/2} \;;\quad f'(z) = \tfrac{1}{2}(z^2-1)^{-1/2}\, 2z = \frac{z}{\sqrt{z^2-1}}$$

(In your homework for next week you will do an analogous procedure for $g(z)=\sqrt{z^3-1}$.) Begin by identifying branch points based on where f or f' cannot be not defined as an analytic function.

f' not defined @ $z=\pm 1$

must be branch pts.

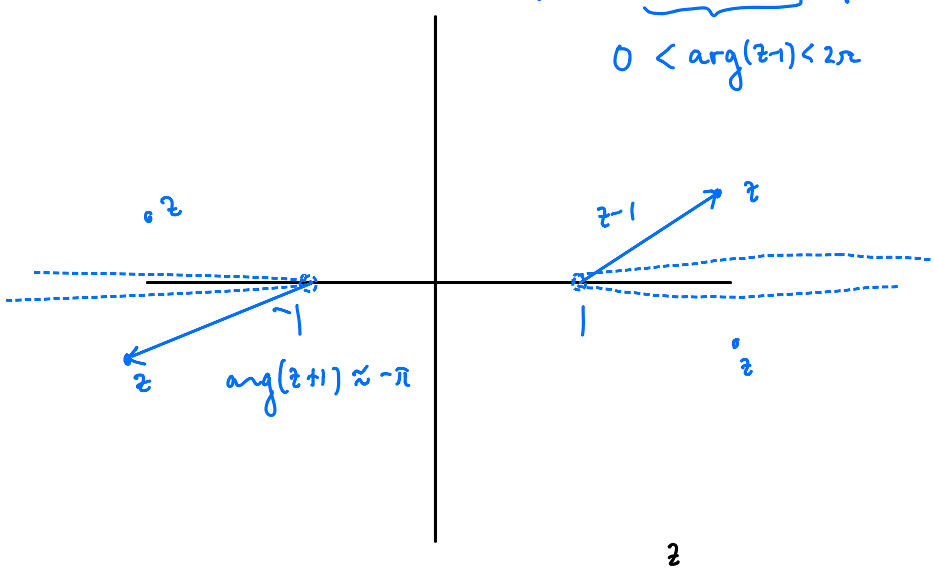
branch domains should open & connected

Then

a) Writing $f(z)=\sqrt{z^2-1}=\sqrt{z-1}\,\sqrt{z+1}$ leads to one possible way of proceding.

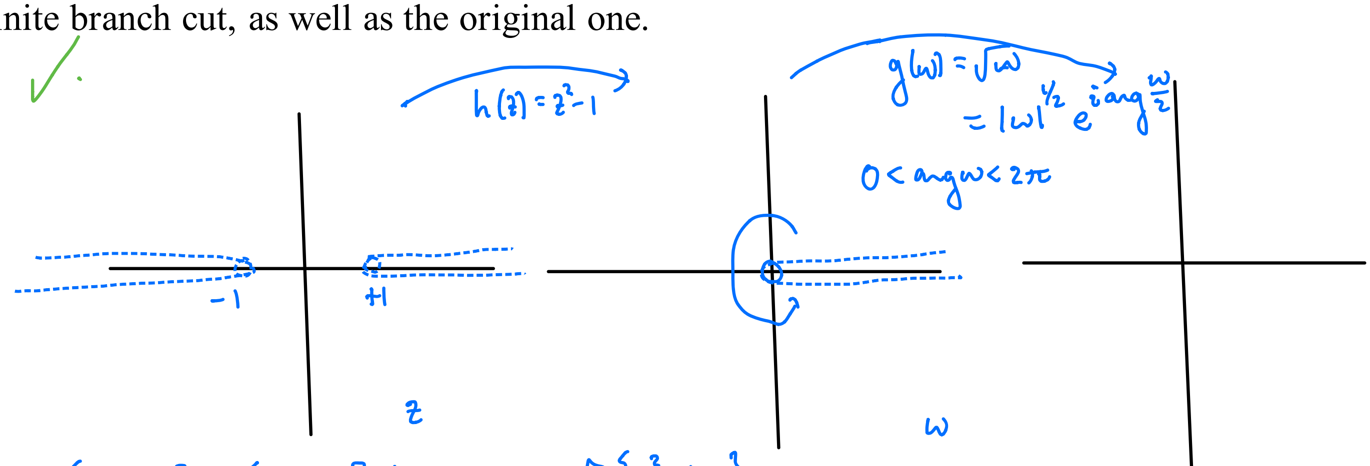
$$= \underbrace{|z-1|^{1/2}\, e^{i\frac{\arg(z-1)}{2}}}_{0<\arg(z-1)<2\pi}\; \underbrace{|z+1|^{1/2}\, e^{i\left(\frac{\arg(z+1)}{2}\right)}}_{-\pi<\arg(z+1)<\pi}$$

is defined on $\mathbb{C} \setminus \{x \in \mathbb{R} \text{ s.t. } |x| \geq 1\}$.



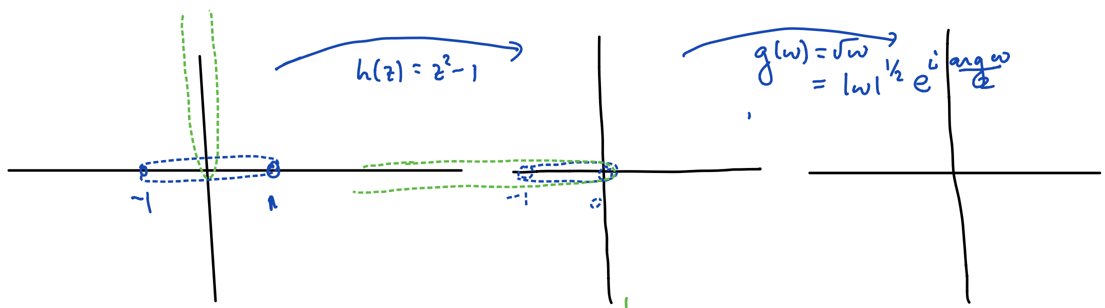
$= \sqrt{z^2-1}$

b) Considering f as a composition, $f(z)=g\circ h(z)$ with $h(z)=z^2-1$ and $g(w)=\sqrt{w}$ recovers the first branched domain, but also leads to a choice with only a finite branch cut, as well as the original one.



$\{x\leq -1\}\cup\{x\geq 1\} \longmapsto \{x^2-1\geq 0\}$.

and nothing else is mapped there since z^2-1 is 2 to 1.



OH NO!

Math 4200
Friday September 18

- 1.6 branched domains (fundamental domains) to create single-valued analytic functions from multi-valued ones. (From Wednesday's notes).
- 2.1 begin Chapter 2 on definite and indefinite complex integration

Chapter 2: Complex integration.
- Leads to Cauchy Integral Formula and magic theorems which result:
    - Liouville's Theorem: Bounded entire functions are constant.
    - Fundamental Theorem of Algebra: every degree n polynomial has n (complex) roots, counting multiplicity.
    - Magic ways to compute many definite integrals (contour integration).

Announcements: HW due on Fridays @ 11:59 pm from now on
HW 4 is in today's notes

2.1 Integration of complex-valued functions of a real variable t, just as in Calc 1. Introduction to *contour integrals* - analogous to *line integrals* from multivariable Calculus.

A1 Def: For $f: [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ continuous, $f(t) = u(t) + i\, v(t)$, with $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$

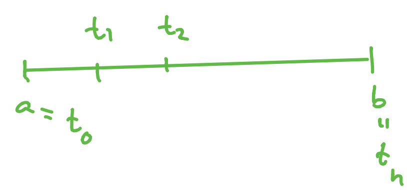
- $$\int_a^b f(t)\ \mathrm{d}t = \int_a^b u(t) + i\, v(t)\ \mathrm{d}t := \int_a^b u(t)\ \mathrm{d}t + i \int_a^b v(t)\ \mathrm{d}t.$$

It is useful for estimates to note that since u, v are continuous on $[a, b]$ they are uniformly continuous - and you proved in Math 3210 that in this case definite integrals are limits of Riemann sums for partionings P of $[a, b]$, as the "norm" of the partition approaches zero: For

$$P := \quad a = t_0 < t_1 < \ldots \ < t_n = b$$

$j=0,\ldots,n-1$ • $t_j \le t_j^* \le t_{j+t}, \quad \Delta t_j = t_{j+1} - t_j$

$$\|P\| := \max \Delta t_j ,$$



3210 $$\int_a^b u(t)\ \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u\left(t_j^*\right)\Delta t_j, \qquad \int_a^b v(t)\ \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j v\left(t_j^*\right)\Delta t_j$$

so also

A2 Def:

- $$\int_a^b f(t)\ \mathrm{d}t = \lim_{\|P\| \to 0} \sum_j u\left(t_j^*\right)\Delta t_j \ + i \lim_{\|P\| \to 0} \sum_j v\left(t_j^*\right)\Delta t_j \ = \lim_{\|P\| \to 0} \sum_j f\left(t_j^*\right)\Delta t_j.$$

Example 1: Use Calc 1 FTC to compute

$$\int_0^{\frac{\pi}{2}} \underbrace{-2 \sin(t)\cos(t)}_{-\sin 2t} + i\big(\underbrace{\cos^2(t) - \sin^2(t)}_{\cos 2t}\big)\ \mathrm{d}t.$$

$$\stackrel{i}{=} \int_0^{\pi/2} -\sin 2t\ dt + i \int_0^{\pi/2} \cos 2t\ dt$$

$$\left.\frac{\cos 2t}{2}\right]_0^{\pi/2} + i \left.\frac{\sin 2t}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\cos\pi - \cos 0\right) + \frac{i}{2}\left(\sin\pi - \sin 0\right)$$

$$= -1 + 0 = -1 \qquad !!$$

Math 4200-001
Week 4-5 concepts and homework
1.6, 2.1-2.2
Due Friday September 25 at 11:59 p.m.

1.6 10, 14
2.1 2ac, 3, 5, 10, 11, 13, 14;
2.2 1ad, 2 (prove with FTC!), 3 (work in reverse to rewrite as a contour integral that you can evaluate), 4, 6, 8, 9, 10 (use the antiderivative theorem and slightly modify Example 1.6.8).

Hint: In many of these problems the fundamental theorem of Calculus for contour integrals lets you find the answer very quickly once you find an antiderivative on an appropriate domain.

w4.1 (extra credit) This is a careful version of 1.6.6. Part (a) is relatively straightforward. I consider part (b) to be challenging.
a) Solve $\sin(z) = w$ for z using the quadratic formula and logarithms. Keep careful track of the multi-valued nature of the inverse sine function $\arcsin(z)$. Note that the quadratic formula yields two solutions except when $\cos(z) = 0$.
b) Prove that there is a branch of $\arcsin(z)$ defined on the branch domain we used in class for $\sqrt{z^2 - 1}$, namely $\mathbb{C} \setminus \{x \in \mathbb{R} \text{ s.t. } |x| \geq 1\}$, which is a bijection to the vertical strip $\left\{x + i\,y \,\middle|\, -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$. This branch extends the Calculus $\arcsin(x)$ which was defined as a differentiable function on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.